- Recall connection
  Idea — way of differentiating sections
- Connection on tensors and total covariant derivative
- Difference of 2 connections
- 2nd fund form for a sub-bundle (and $\exists$ of connections)
- Properties of the tangential connection:
  - metric
  - symmetric

Consequences:

(1) commutes w/ raising & lowering

$$\nabla_i X^j = g^{jk} \nabla_i X_k$$

(2) If θ_i is a 1-form,

then $\nabla\theta$ symmetric $\Longleftrightarrow$ θ closed.

$$\ulcorner \; (\nabla_X \theta)(Y) = X(\theta(Y)) - \theta(\nabla_X Y) \; \lrcorner$$

- Thm $\exists!$ of L-C connection

$$\ulcorner \; X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$$= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X,Z]\rangle$$

+ cyclic
add 2
subtract third

cor: • $\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$

- If $[E_i, E_j] = E_k c_{ij}^k$

then

$$\Gamma_{ij}^k = \frac{1}{2} \left(c_{ij}^k - c_{ik}^j - c_{jk}^i \right)$$

$\lrcorner$

Geometry

$S \subseteq \mathbb{R}^3$

$\alpha : I \to S$ a geodesic if $k_g = \langle T', U \rangle = 0$

i.e. $\pi^{T} \overline{\nabla}_{\partial_t} \alpha' = 0$ (assuming unit speed)

i.e. $\nabla_{\partial_t} \alpha' = 0$

connection on TM

Def'n $\alpha : I \to (M, \nabla)$ is a geodesic if $\nabla_t \alpha' = 0$.

Observation
- geodesics determine $\Gamma_a{}^c{}_b T^a T^b \leftrightarrow$ symmetric part of Γ
- do not determine skew-symm. part

Now suppose ∇ is $\nabla^{L.C.}$

Dfn A smooth map $\alpha(s)(t) : (-\varepsilon, \varepsilon) \times I \to S$ is a compactly supported variation of the curve $\alpha(0)$ if

- $\alpha(s)$ is a regular parametrized curve in S for all $s \in (-\varepsilon, \varepsilon)$
- outside some $K \subset\subset I$, $\alpha(s)(t) = \alpha(0)(t)$

Prop Geodesics in S are critical points of the length with respect to compact variations

$\langle \nabla_s \alpha_t, T \rangle$

$\langle \nabla_t \alpha_s, T \rangle$

Pf $\frac{d}{ds} \int_I \sqrt{\langle \alpha_t, \alpha_t \rangle}\, dt = \int_I \frac{\langle \alpha_{ts}, \alpha_t \rangle}{|\alpha_t|} dt = \int_I \langle \alpha_{st}, T \rangle dt$

"$\frac{d}{ds} L(\alpha(s))$"

$= \int_I \frac{d}{dt} \langle \alpha_s, T \rangle dt - \int_I \langle \alpha_s, T' \rangle dt$

0

Since $\alpha_s \in TS$ for every variation, this is zero for every variation exactly when T' is parallel to N, i.e. $\langle T', U \rangle = 0$

i.e. $k_g = 0$ $\dashv$

- Connections and ODEs

(G, ∇), $\quad F = [F_1 \cdots F_n]$ a frame $\quad F : \mathbb{R}^n \to G$,

let $\nabla F_i = F_j A^j_i$ be the coeffs of the connection in this frame

$$\nabla(F_i f^i) = F_i(df^i + A^i_j f^j)$$

$$F^* \nabla = d + A$$

A second frame $\tilde{F}_j = F_i (G^{-1})^i_j$ is flat if

$$dG^{-1} + AG^{-1} = 0.$$

$$-G^{-1} dG\, G^{-1} + AG^{-1} = 0$$

$$A = G^{-1} dG$$

- Thm (Generalizing last time) A flat frame exists if $dA + A \wedge A = 0$.

- Returning to a surface in $\mathbb{R}^3$
  (1) $\mathbb{R}^3 = TS + NS$
  (2) Torsion-free related to integrating all the way to $\mathbb{R}^3$, which we ignored before.

- Hessian + convexity
- Geodesics + parallel transport
- The geodesic equation

Dfn The curvature of ∇ $\quad R \in \Omega^2(\mathrm{End}\, E)$

$$R_{ab} e = \nabla_a \nabla_b e - \nabla_b \nabla_a e .$$

Calc curvature of $d + A$ on triv bundle is

$$R = dA + A \wedge A .$$

Generalization of last time to n dimensions $\Longrightarrow$

Thm $\exists\, F$ w/ $F^{-1} dF = A$ iff $R_A = 0$.

i.e.

Thm (E, ∇) has a local basis of flat sections $\Leftrightarrow R^\nabla = 0$.

Returning to a surface in $\mathbb{R}^3$

$(\mathbb{R}^3, \overline{\nabla}) = TS + NS.$

$$\overline{\nabla}_X Y = \nabla_X Y + B_X Y$$

$$\overline{R} = 0$$

in local frame for TS + triv frame for $\langle N \rangle$, $\overline{\nabla} = d + \overline{\Gamma}$ where

$$\overline{\Gamma} = \begin{bmatrix} \Gamma & -B \\ h & 0 \end{bmatrix}$$

$B \in \Omega^1(\mathrm{Hom}(N,T)) \cong \mathrm{End}(T)$

$h \in \Omega^1(\mathrm{Hom}(T,N)) \cong \otimes^2 T^*$

$B \wedge h \in \Omega^2(\mathrm{Hom}(T,T))$

$$d\overline{\Gamma} + \overline{\Gamma} \wedge \overline{\Gamma} = \begin{bmatrix} d\Gamma + \Gamma \wedge \Gamma - B \wedge h & -dB - \Gamma \wedge B \\ dh + h \wedge \Gamma & 0 \end{bmatrix}$$

$$= 0$$

(Gauss + Codazzi equations)

Integrable to a frame. $\Longleftrightarrow$ $dA + A \wedge A = 0$

But I skipped: when is a frame $F : TM \to \mathbb{R}^3$ 'integrable to a surface?

Require $dF = 0$ i.e.

$$\bar{\nabla}_X(F(Y)) - \bar{\nabla}_Y(F(X)) - F([X,Y]) = 0$$

i.e. $\pi^N = 0$ (1) and $\pi^T = 0$ (2)

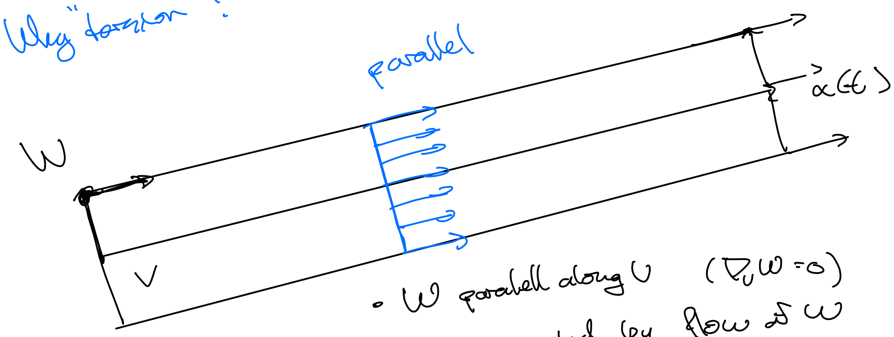
$$A^{\bullet}_{12} = A^{\bullet}_{21}$$

(1) $h(X,Y) = h(Y,X)$

(2) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$

Torsion

Why "torsion"?



- W parallel along V $\quad (\nabla_V W = 0)$
- V transported by flow of W $\quad ([V, W] = 0)$

$\exists\ \psi(t,s)$ s.t. $\dfrac{\partial \psi}{\partial t} = V$, $\quad \dfrac{\partial \psi}{\partial s} = W$

$$\Rightarrow \nabla_W V = 0$$

no spiralling

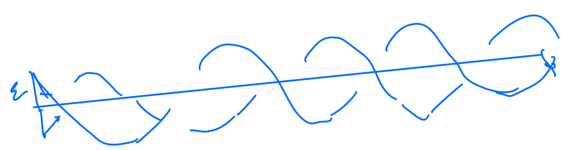
$$\frac{\partial}{\partial s} \langle \psi_t, \psi_t \rangle$$

Let's suppose $\nabla_V V = 0$, so $W \perp V$ always

same order as curvature

as opposed to

$\psi(\varepsilon, \theta) = \begin{matrix} \varepsilon \cos\theta \\ \varepsilon \sin\theta \\ \theta \end{matrix}$ $\quad |\psi'| = \sqrt{1 - c\varepsilon^2}$



Apparently $\langle \nabla_W V, V \rangle$ is independent

"looks like" is weird tho ______

if you walk along V, W looks parallel.